Correction

Teacher : Emmanuel Abbe
MATH-232 Midterm Exam -
7 November 2024
75 minutes

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Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 12 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- First part, for the **single choice** questions, we give :

  $+3$ points if your answer is correct,
  0 points if you give no answer or more than one,
  -1 points if your answer is incorrect.

- Second part, for the **true/false** questions, we give :

  $+1$ points if your answer is correct,
  0 points if you give no answer or more than one,
  -1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Correction

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has exactly one correct answer.

Question 1 Let $X \sim \mathcal{N}(3, 4)$ follow a Normal distribution. Define $Y = 2X - 1$. What is the closest number to $\mathbb{P}(Y > 6)$?

- ■ 0.4
- □ 0.3
- □ 0.7
- □ 0.6
- □ 0.5

Question 2 A system's lifetime follows an exponential distribution with a mean of 5 years. If the system has already been operating for 3 years without failure, what is the expected remaining lifetime of the system?

- ■ 5 years
- □ 3 years
- □ None of the options
- □ 2 years

Question 3 Suppose X, Y, Z are three random variables where $Z = X + 2Y$. Let $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$ denote the variance of X, Y, and Z, respectively. Consider the following claims:

$$\text{Claim 1: } \sigma_Z^2 \leq \sigma_X^2 + 4\sigma_Y^2$$
$$\text{Claim 2: } \sigma_Z \leq \sigma_X + 2\sigma_Y$$

Which one of the claims is **always** correct?

- □ Both claims
- □ None of the claims
- ■ Only claim 2
- □ Only claim 1

Question 4 There are 2 drawers. The first drawer contains only black balls. The second contains 50% black balls and 50% white balls. There are an equal number of balls in each drawer. We pick a ball at random and it is black. What is the probability that the ball came from the first drawer?

- □ $\frac{1}{2}$
- □ $\frac{1}{4}$
- ■ $\frac{2}{3}$
- □ $\frac{3}{4}$

Question 5 Let X be a random variable that follows a uniform distribution on the interval $[-1, 1]$, i.e., $X \sim \text{Uniform}(-1, 1)$. Define a new random variable $Y = X^2$. Which of the following is the probability density function (PDF) of Y?

- ■ $f_Y(y) = \frac{1}{2\sqrt{y}}$, for $0 \leq y \leq 1$
- □ $f_Y(y) = \frac{1}{4\sqrt{y}}$, for $0 \leq y \leq 1$
- □ $f_Y(y) = 1$, for $0 \leq y \leq 1$
- □ $f_Y(y) = \sqrt{y}$, for $0 \leq y \leq 1$

Question 6 Let $x \in \mathbb{R}$. Which of the following can be a valid CDF (Cumulative Distribution Function)?

- ■ $F(x) = \exp(-e^{-x})$
- □ $F(x) = 1 - \exp(-1/|x|)$
- □ $F(x) = 1 - \exp(x)$
- □ $F(x) = \frac{1}{1-e^{-x}}$

Question 7 In an NBA basketball finals game between team A, B, the winner is the team who wins 4 out of a maximum of 7 games. What is the probability of the finals getting to the 7^{th} game, assuming both teams are equally likely to win a single game?

- ■ 0.3125
- □ 0.375
- □ 0.5
- □ 0.25
- □ 0.625

Question 8 Consider a multinomial distribution with parameters n (number of trials) and $p_1, p_2, \ldots, p_k$ (probabilities of each outcome). Let $X_1, X_2, \ldots, X_k$ represent the number of occurrences of each outcome. What is the value of $Cov(X_i, X_j)$ for $i \neq j$ss?

- □ np_ip_j for $i \neq j$
- □ p_ip_j for $i \neq j$
- □ $n^2p_ip_j$ for $i \neq j$
- □ $-n^2p_ip_j$ for $i \neq j$
- □ $-p_ip_j$ for $i \neq j$
- ■ $-np_ip_j$ for $i \neq j$

Question 9 Both A and C are infected, and the probability of transmitting the disease to B is 0.5 from A and 0.3 from C. B meets first with A and then with C. If B has been infected, what is the probability that B was infected by A (assuming that if B is infected by A, C cannot further transmit the disease)?

- □ $\frac{9}{14}$
- □ None of the options
- □ $\frac{10}{15}$
- ■ $\frac{10}{13}$
- □ $\frac{11}{15}$
- □ $\frac{11}{14}$

Question 10 Let $U \sim \text{Uniform}(0, 1)$ and define $X = 4\ln(1/U)$. What is $\mathbb{E}[X]$ equal to:

- ■ 4
- □ 8
- □ 1
- □ ∞
- □ 2

Question 11 Let the probability density function of the random variable X be

$$f(x) = \frac{3}{8}x^2, \quad 0 \leq x \leq 2.$$

What is the expectation of $\frac{1}{X}$?

- ☐ $\frac{3}{2}$
- ☐ 1
- ☐ $\frac{2}{3}$
- ☐ $\frac{3}{8}$
- ■ $\frac{3}{4}$

Question 12 Tomy is angry by default. However, he becomes kind if he receives at least two hugs within one day. As usual, we model the probabilities of rare events using Poisson distribution. Suppose the number of hugs Tomy receives each day follows Poisson distribution with parameter λ. What is the probability that Tomy has experienced a state of kindness at least once during the last week?

- ☐ $7(1-(\lambda+1)e^{-\lambda})$
- ☐ $1-\lambda^7 e^{-7\lambda}$
- ☐ $(1-\lambda e^{-\lambda})^7$
- ☐ None of the options
- ■ $1-(\lambda+1)^7 e^{-7\lambda}$
- ☐ $(1-(\lambda+1)e^{-\lambda})^7$

Question 13 Aubameyang, Bukayo, and Calafiori take turns in alphabetical order (so ABCABCABC...) throwing a fair coin until one of them gets tails. Whomever get tails first wins. Which one is the winning probability of Bukayo?

- ☐ $\frac{2}{5}$
- ☐ $\frac{1}{4}$
- ■ $\frac{2}{7}$
- ☐ $\frac{1}{3}$
- ☐ None of the options

Question 14 A mathematician forgets his umbrella in a store with a probability of $\frac{1}{4}$ (independtly from the store); today, he went shopping in four shops. Knowing that he went home without his umbrella (i.e. he forgot it in one of the stores), what is the probability that he forgot it in the last shop.

- ☐ $\frac{27}{256}$
- ☐ $\frac{1}{4}$
- ☐ $\frac{3}{16}$
- ■ $\frac{27}{175}$
- ☐ $\frac{9}{64}$

Second part: true/false questions

For each question, mark the box TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 15 Let A, B be the events of some probability space. The independence of A and B always imply the independence of A^c and B^c.

■ TRUE ☐ FALSE

Basic formulas and definitions

- Properties of binomial coefficients
  (a) Pascal's triangle $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
  (b) Vandermonde's formula $\sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j} = \binom{m+n}{r}$.
  (c) Negative binomial series $(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$, $|x| < 1$.
  (d) $\lim_{n\to\infty} n^{-r}\binom{n}{r} = \frac{1}{r!}$, where $r \in \mathbb{N}$ is fixed.
- $\exp(x) = e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.
- Inclusion-exclusion formula:

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1\le i_1<i_2<\ldots<i_r\le n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r})$$

- Gamma function is defined as $\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} e^{-u} du$ for $\alpha > 0$. It further satisfies:
  - $\Gamma(n) = (n-1)! \quad n \in \mathbb{N}$.
  - $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \alpha > 0$.
- Let $g(X, Y)$ be a function of a random vector (X, Y). Its conditional expectation given $X = x$ is

$$\mathbb{E}[g(X,Y)|X=x] = \begin{cases} \sum_y g(x,y) f_{Y|X}(y|x) & \text{discrete case} \\ \int g(x,y) f_{Y|X}(y|x) dy & \text{continuous case} \end{cases}$$

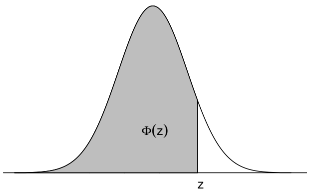
  on the condition that $f_X(x) > 0$ and $\mathbb{E}[|g(X,Y)||X=x] < \infty$.
- For random variables X and $Y = g(X)$, where g is a monotone increasing or decreasing function with differentiable inverse g^{-1}, we have

$$f_Y(y) = |\frac{dg^{-1}(y)}{dy}| f_X(g^{-1}(y)).$$

Distributions

Distribution	PMF/PDF	Expected Value	Variance	MGF
Bernoulli Bern(p)	$P(X=1)=p$ $P(X=0)=1-p$	p	$p(1-p)$	$1-p+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$ $k \in \{0,1,2,\dots n\}$	np	$np(1-p)$	$(1-p+pe^t)^n$
Geometric Geom(p)	$P(X=k)=(1-p)^{k-1}p$ $k \in \{1,\ 2,\ \dots\}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$ $(1-p)e^t<1$
Neg. Binom. NegBin(r,p)	$P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$ $x \in \{r,r+1,r+2,\dots\}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$(\frac{pe^t}{1-(1-p)e^t})^r$ $(1-p)e^t<1$
Hypergeom. HypG(w,b,n)	$P(X=k)=\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$ $k \in \{0,1,2,\dots,n\}$	$\mu=\frac{nw}{b+w}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}(1-\frac{\mu}{n})$	messy
Poisson Pois(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k \in \{0,\ 1,\ 2,\ \dots\}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform U(a,b)	$f(x)=\frac{1}{b-a}$ $x \in [a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential exp(λ)	$f(x)=\lambda e^{-\lambda x}$ $x \in (0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x \in (-\infty,\infty)$	μ	σ^2	$e^{t\mu+\frac{\sigma^2t^2}{2}}$
Chi-Square χ^2_n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$ $x \in (0,\infty)$	n	$2n$	$(1-2t)^{-n/2}$ $t<1/2$
Gamma $\Gamma(\alpha,\lambda)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$ $x \in (0,\infty)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$(1-\frac{t}{\lambda})^{-\alpha}$ $t<\lambda$

Standard normal distribution Φ(z)



For $z < 0$ we use symmetry: $\mathbb{P}(Z \leq z) = \Phi(z) = 1 - \Phi(-z)$, $z \in \mathbb{R}$.

z	0	1	2	3	4	5	6	7	8	9
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56750	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84850	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92786	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

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